

Second order Lax pairs of nonlinear partial differential equations with Schwarz variants

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Abstract

In this paper, we study the possible second order Lax operators for all the possible (1+1)-dimensional models with Schwarz variants and some special types of high dimensional models. It is shown that for every (1+1)-dimensional model and some special types of high dimensional models which possess Schwarz variants may have a second order Lax pair. The explicit Lax pairs for (1+1)-dimensional Korteweg de Vries equation, Harry Dym equation, Boussinesq equation, Caudry-Dodd-Gibbon-Sawada-Kortera equation, Kaup-Kupershmidt equation, Riccati equation, (2+1)-dimensional breaking soliton equation and a generalized (2+1)-dimensional fifth order equation are given.

1 Introduction

In the study of a nonlinear mathematical physics system, if one can find that the nonlinear system can be considered as a consistent condition of a pair of a linear problem, then some types of special important exact solutions of the nonlinear system can be solved by means of the pair of linear problem. The pair of linear system is called as Lax pair of the original nonlinear system and the nonlinear system is called as Lax integrable or IST (inverse scattering transformation) integrable. Usually, a Lax integrable model may have also many other interesting properties, like the existing of infinitely many conservation laws and infinitely many symmetries,

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multi-soliton solutions, bilinear form, Schwarz variants, multi-Hamiltonian structures, Painlevé property etc. In the recent studies, we found that the existence of the Schwarz variants may play an important role. Actually, in our knowledge, almost all the known IST integrable (1+1)- and (2+1)-dimensional models possess Schwarz invariant forms which are invariant under the Möbius transformation (conformal invariance)[1, 2]. The conformal invariance of the well known Schwarz Korteweg de-Vries (SKdV) equation is related to the infinitely many symmetries of the usual KdV equation[3]. The conformal invariant related flow equation of the SKdV is linked with some types of (1+1)-dimensional and (2+1)-dimensional sinh-Gordon (ShG) equations and Mikhailov-Dodd-Bullough (MDB) equations[4]. It is also known that by means of the Schwarz forms of many known integrable models, one can find also many other integrable properties like the Bäcklund transformations and Lax pairs[1]. In [5], one of the present authors (Lou) proposed that starting from a conformal invariant form may be one of the best ways to find integrable models especially in high dimensions. Some types of quite general Schwarz equations are proved to be Painlevé integrable. In [6], Conte's conformal invariant Painlevé analysis[7] is extended to obtain high dimensional Painlevé integrable Schwarz equations systematically. And some types of physically important high dimensional nonintegrable models can be solved approximately via some high dimensional Painlevé integrable Schwarz equations[8].

Now an important question is what kind of Schwarz equations are related to some Lax integrable models? To answer this question generally in arbitrary dimensions is still quite difficult. So in this paper we restrict our main interests to discuss in (1+1)-dimensional models.

In the next section, we prove that for any (1+1)-dimensional Schwarz model there may be a second order Lax pair linked with it. In section 3, we list various concrete physically significant examples. In section 4, we discuss some special extensions in higher dimensions. The last section is a short summary and discussion.

2 A second order (1+1)-dimensional Lax pair Linked with an arbitrary Schwarz form

In (1+1)-dimensions, the only known independent conformal invariants are

$$p_1 \equiv \frac{\phi_t}{\phi_x}, \quad (1)$$

$$p_2 \equiv \{\phi; x\} \equiv \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2} \frac{\phi_{xx}^2}{\phi_x^2}, \quad (2)$$

$$p_3 \equiv \{\phi; t\} \equiv \frac{\phi_{ttt}}{\phi_t} - \frac{3}{2} \frac{\phi_{tt}^2}{\phi_t^2}, \quad (3)$$

$$p_4 \equiv \{\phi; x; t\} \equiv \frac{\phi_{xxt}}{\phi_t} - \frac{\phi_{xx}\phi_{xt}}{\phi_x\phi_t} - \frac{1}{2} \frac{\phi_{xt}^2}{\phi_t^2}, \quad (4)$$

$$p_5 \equiv \{\phi; t; x\} \equiv \frac{\phi_{xtt}}{\phi_x} - \frac{\phi_{tt}\phi_{xt}}{\phi_x\phi_t} - \frac{1}{2} \frac{\phi_{xt}^2}{\phi_x^2}, \quad (5)$$

where ϕ is a function of $\{x, t\}$, the subscripts are usual derivatives while $\{\phi; x\}$ is the Schwarz derivative. As in [5, 6, 8], we say a quantity is a conformal invariant if it is invariant under the Möbius transformation

$$\phi \rightarrow \frac{a\phi + b}{c\phi + d}, \quad ad \neq bc. \quad (6)$$

From (1)–(5), we know that the general (1+1)-dimensional conformal invariant Schwarz equation has the form

$$F(x, t, p_i, p_{ix}, p_{it}, p_{ixx}, \dots (i = 1, \dots, 5)) \equiv F(p_1, p_2, p_3, p_4, p_5) = 0, \quad (7)$$

where F may be an arbitrary function of x, t, p_i and any order of derivatives and even integrations of p_i with respect to x and t . According to the idea of [5], (7) (or many of (7)) may be integrable. If F of (7) is a polynomial function of p_i and the derivatives of p_i , then one may prove its Painlevé integrability by using the method of [5, 6]. However, for general function F in (7), it is difficult to prove its Painlevé integrability. Fortunately, we can find its relevant variant forms with Lax pair. To realize this idea, we consider the following second order Lax pair:

$$\psi_{xx} = u\psi_x + v\psi, \quad (8)$$

$$\psi_t = u_1\psi_x + v_1\psi, \quad (9)$$

where u, u_1, v , and v_1 are undetermined functions. To link the Lax pair (8) and (9) with the Schwarz equation (7), we suppose that ψ_1 and ψ_2 are two solutions of (8) and (9), and ϕ of (7) is linked to ψ_1 and ψ_2 by

$$\phi = \frac{\psi_1}{\psi_2}. \quad (10)$$

Now by substituting (10) with (8) and (9) into (7) directly, we know that if the functions u, v and u_1 are linked by

$$F(P_1, P_2, P_3, P_4, P_5) = 0 \quad (11)$$

with

$$P_1 = u_1, \quad P_2 = u_x - \frac{1}{2}u^2 - 2v, \quad (12)$$

$$P_3 = P_2u_1^2 + u_1u_{1xx} - \frac{1}{2}u_{1x}^2 + u_{1xt} + u_1^{-1}(u_{1tt} - u_{1x}u_{1t}) - \frac{3}{2}u_1^{-2}u_{1t}^2, \quad (13)$$

$$P_4 = P_2 + u_{1xx}u_1^{-1} - \frac{1}{2}u_1^{-2}u_{1x}^2, \quad (14)$$

$$P_5 = u_1^2P_4 + u_{1xt} - u_1^{-1}u_{1t}u_{1x}, \quad (15)$$

then the corresponding nonlinear equation system for the fields u , v , u_1 and v_1 has a Lax pair (8) and (9) while the fields u , v , u_1 and v_1 are linked to the field ϕ by the non-auto-Bäcklund transformation

$$p_i = P_i, (i = 1, 2, \dots, 5). \quad (16)$$

Finally to find the evolution equation system is a straightforward work by calculating the compatibility condition of (8) and (9),

$$\psi_{xxt} = \psi_{txx}. \quad (17)$$

The result reads

$$v_t = v_{1xx} + 2vu_{1x} + u_1v_x - uv_{1x}, \quad (18)$$

and

$$u_t = u_{1xx} + 2v_{1x} + (uu_1)_x \quad (19)$$

in addition to the constraint (11). In Eqs. (11), (18) and (19), one of four functions u , u_1 , v , and v_1 remains still free. For simplicity, one can simply take

$$u = 0, \quad v_1 = -\frac{1}{2}u_{1x}. \quad (20)$$

Under the simplification (20), the final evolution equation related to the Schwarz form (7) read

$$v_t = -\frac{1}{2}u_{1xxx} + 2vu_{1x} + u_1v_x \quad (21)$$

with (11) for $u = 0$ while the Lax pair is simplified to

$$L\psi \equiv (\partial_x^2 - v)\psi = 0, \quad (22)$$

$$\psi_t = M\psi \equiv (u_1\partial_x - \frac{1}{2}u_{1x})\psi. \quad (23)$$

It should be emphasized again that the Lax operator given in (22) is only a second order operator.

To see the results more concretely, we discuss some special physically significant models in the following section.

3 Special examples

From the suitable selections of $F \equiv F(p_1, p_2, p_3, p_4, p_5)$ of (7), we may obtain various interesting examples according to the general theory of the last section.

Example 1. KdV equation.

For the KdV equation, its Schwarz variant has the simple form

$$F_{KdV}(p_i) = p_1 + p_2 = 0. \quad (24)$$

According to the formula (11) with $u = 0$, we know that the relation between the functions v and u_1 is simply given by

$$v = \frac{1}{2}u_1. \quad (25)$$

Substituting (25) into (22) and (23), we re-obtain the well known Lax pair

$$\psi_{xx} - \frac{1}{2}u_1\psi = 0, \quad (26)$$

$$\psi_t = u_1\psi_x - \frac{1}{2}u_{1x}\psi \quad (27)$$

for the KdV equation

$$u_{1t} = 3u_1u_{1x} - u_{1xxx}. \quad (28)$$

Example 2. Harry-Dym (HD) equation.

For the HD equation, the Schwarz form reads

$$F_{HD}(p_i) = p_1^2 - \frac{2}{p_2} = 0 \quad (29)$$

which leads to the relation between the functions v and u_1 by

$$v = \frac{1}{u_1^2}. \quad (30)$$

From (22), (23) and (30), one can obtain the known Lax pair

$$\psi_{xx} - \frac{1}{u_1^2}\psi = 0, \quad (31)$$

$$\psi_t = u_1\psi_x - \frac{1}{2}u_{1x}\psi \quad (32)$$

for the HD equation

$$u_{1t} = \frac{1}{4}u_1^3u_{1xxx}. \quad (33)$$

Example 3. Modified Boussinesq Equation and Boussinesq equation

For the modified Boussinesq (MBQ) equation (and the Boussinesq equation), the Schwarz form has the form

$$F_{MBQ}(p_i) = p_{2x} + 3p_1p_{1x} + 3p_{1t} = 0. \quad (34)$$

Using (34) and (11), we have

$$v = \frac{3}{4}u_1^2 + \frac{3}{2} \int u_{1t} dx. \quad (35)$$

Substituting (35) into (22) and (23) we get a Lax pair

$$\psi_{xx} - \left(\frac{3}{4}u_1^2 + \frac{3}{2} \int u_{1t} dx \right) \psi = 0, \quad (36)$$

$$\psi_t = u_1 \psi_x - \frac{1}{2} u_{1x} \psi. \quad (37)$$

The related compatibility condition of (36) and (37) reads

$$3u_1^2 u_{1x} + 3u_{1x} \int u_{1t} dx - \frac{1}{2} u_{1xxx} - \frac{3}{2} \int u_{1tt} dx = 0. \quad (38)$$

Eq. (38) is called as the modified Boussinesq equation because it is linked with the known Boussinesq equation

$$u_{tt} + \left(3u^2 + \frac{1}{3} u_{xx} \right)_{xx} = 0 \quad (39)$$

by the Miura transformation

$$u = \frac{1}{3} (\pm u_{1x} - u_1^2 - \int u_{1t} dx). \quad (40)$$

Example 4. Generalized fifth order KdV (FOKdV) equation.

The generalized fifth order Schwartz KdV equation has the form

$$F_{FOKdV}(p_i) = p_1 - a_1 p_{2xx} - a_2 p_2^2 = 0, \quad (41)$$

where a_1 and a_2 are arbitrary constants. Using (41) and (11), we have

$$u_1 = -2a_1 v_{xx} + 4a_2 v^2. \quad (42)$$

Substituting (42) into (22) and (23) we get

$$\psi_{xx} - v\psi = 0, \quad (43)$$

$$\psi_t = (a_1 v_{xxx} - 4a_2 v v_x) \psi - 2(a_1 v_{xx} - 2a_2 v^2) \psi_x. \quad (44)$$

The related compatibility condition of (43) and (44) is the generalized FOKdV equation

$$v_t - a_1 v_{xxxxx} + 4(a_1 + a_2) v v_{xxx} + 2(a_1 + 6a_2) v_x v_{xx} - 20a_2 v^2 v_x = 0. \quad (45)$$

Some well known fifth order integrable partial differential equations are just the special cases of (45). The usual FOKdV equation is related to (45) for

$$a_1 = 1, \quad a_2 = \frac{3}{2}. \quad (46)$$

The Caudry-Dodd-Gibbon-Sawada-Kortera equation is related to (45) for

$$a_1 = 1, \quad a_2 = \frac{1}{4} \quad (47)$$

while the parameters a_1 and a_2 for the Kaup-Kupershmidt equation read

$$a_1 = 1, \quad a_2 = 4. \quad (48)$$

Example 5. Generalized seventh order KdV (SOKdV) equation.

The generalized seventh order Schwartz KdV equation has the form

$$F_{SOKdV}(p_i) = p_1 - p_{2xxxx} - \alpha p_2 p_{2xx} - \beta p_{2x}^2 - \lambda p_2^3 = 0, \quad (49)$$

where α , β and λ are arbitrary constants. Using (49) and (11), we have

$$u_1 = -2v_{xxxx} + 4\alpha v v_{xx} + 4\beta v_x^2 - 8\lambda v^3. \quad (50)$$

Substituting (50) into (22) and (23) we get

$$\psi_{xx} - v\psi = 0, \quad (51)$$

$$\begin{aligned} \psi_t = & (v_{xxxxx} - 2(\alpha + 2\beta)v_x v_{xx} - 2\alpha v v_{xxx} + 12\lambda v^2 v_x)\psi \\ & + (-2v_{xxxx} + 4\alpha v v_{xx} + 4\beta v_x^2 - 8\lambda v^3)\psi_x. \end{aligned} \quad (52)$$

The related compatibility condition of (50) and (51) is the generalized SOKdV equation

$$\begin{aligned} v_t - v_{xxxxxx} + 2(\alpha + 2\beta)v v_{xxxxx} + 2(1 + 2\beta + 3\alpha)v_x v_{xxx} - (16\beta + 12\alpha + 72\lambda)v v_x v_{xx} \\ + (8\alpha v_{xx} - 8\alpha v^2 + 4\beta v_{xx} - 12\lambda v^2)v_{xxx} - (24\lambda + 4\beta)v_x^3 + 56\lambda v^3 v_x = 0. \end{aligned} \quad (53)$$

The usual SOKdV equation is related to (53) for

$$\alpha = 5, \quad \beta = \frac{5}{2}, \quad \lambda = \frac{5}{2}. \quad (54)$$

The seventh order CDGSK equation corresponds to

$$\alpha = 12, \quad \beta = 6, \quad \lambda = \frac{32}{3}. \quad (55)$$

The parameters of the seventh order KK equation can be read from

$$\alpha = \frac{3}{2}, \quad \beta = \frac{3}{4}, \quad \lambda = \frac{1}{6}. \quad (56)$$

Example 6. Riccati equation (RE)

If the Schwarz form (7) is simply taken as

$$F_{SKdV}(p_i) \equiv p_5 = 0, \quad (57)$$

then we have

$$v = \frac{1}{2}u_1^{-1}u_{1xx} + \frac{1}{4}u_1^{-2}(2u_{1xt} - u_{1x}^2) - \frac{1}{2}u_1^{-3}u_{1t}u_{1x}. \quad (58)$$

The evolution equation of u_1 reads

$$3u_1u_{1t}u_{1xt} - u_1^2u_{1xtt} + u_1u_{1xt}u_{1tt} - 3u_{1x}u_{1t}^2 = 0, \quad (59)$$

while the related Lax pair reads

$$\psi_{xx} - \left(\frac{1}{2}u_1^{-1}u_{1xx} + \frac{1}{4}u_1^{-2}(2u_{1xt} - u_{1x}^2) - \frac{1}{2}u_1^{-3}u_{1t}u_{1x}\right)\psi = 0, \quad (60)$$

$$\psi_t = \left(-\frac{1}{2}u_{1x} + \lambda_1\right)\psi + u_1\psi_x. \quad (61)$$

Actually (59) is equivalent to a trivial linearizable Riccati equation

$$w_t = w^2 + f_1(x) \quad (62)$$

under the transformation

$$u_1 = \exp\left(2 \int w dt\right) \quad (63)$$

where $f_1(x)$ is an arbitrary function of x . It is worth to mention again that the well known (1+1)-dimensional ShG model and MDB model are just the non-invertible Miura type deformation of the Riccati equation[4].

4 Special extensions in higher dimensions

From section 2, we know that the key procedure to find Lax pair from the general conformal invariant form (7) is to find a suitable Lax form ansatz (like (8) and (9)) and a suitable relation ansatz (like (10) between the field of the Schwarz form and the spectral function such that all the conformal invariants (p_i) becomes spectral function independent variables (P_i).

To extend this idea to high dimension is quite not easy. We hope to solve this problem in future studies. In this section we give out some special extensions in high dimensions with the same Lax pair forms of (8) and (9).

If all the fields are functions of not only $\{x, t\}$, but also $\{y, z, \dots\}$ then all the formal theory is still valid if the independent conformal invariants of (11) is still restricted as p_i , $i = 1, \dots, 5$ while the function F of (11) may also include some derivatives and integrations of p_i with respect to other space variables y, z, \dots etc. Here we list only two special examples:

Example 7. (2+1)-dimensional KdV type breaking soliton equation

The concept of breaking soliton equations is firstly developed by [9] and [10] by extending the usual constant spectral problem to non-constant spectral problem. Various interesting properties of the breaking soliton equations have been revealed by many authors. For instance, infinitely many symmetries of some breaking soliton equations are given in [11, 12]. In [13], it is pointed out that every (1+1)-dimensional integrable model can be extended to some higher dimensional

breaking soliton equations with help of its strong symmetries. Yu and Toda [14] had given out the Schwarz form of the (2+1)-dimensional KdV type breaking soliton equation

$$F_{2dSKdV} \equiv p_1 + \int p_{2y} dx = 0. \quad (64)$$

From (11) and (64), we have

$$u_1 = 2 \int v_y dx. \quad (65)$$

Substituting (65) into (22) and (23), we obtain a Lax pair

$$\psi_{xx} = v\psi, \quad (66)$$

$$\psi_t = 2 \int v_y dx \psi_x - (v_y - \lambda_1)\psi. \quad (67)$$

for the (2+1)-dimensional KdV type breaking soliton equation

$$v_t = -v_{xxy} + 4vv_y + 2v_x \int v_y dx \equiv \Phi v_y, \quad (68)$$

where Φ is just the strong symmetry of the (1+1)-dimensional KdV equation.

Example 8. (2+1)-dimensional fifth order equation

If we make the replacement

$$p_2 \rightarrow \int p_{2y} dx \quad (69)$$

for some of p_2 in all the examples of the Last section, then we can obtain some special types of their (2+1)-dimensional extensions. Example 8 is just obtained from the (1+1)-dimensional KdV equation by using the replacement (69).

A generalization of the fifth order Schwarz equation (41) reads ($b_1 + b_2 = a_1$, $c_1 + c_2 + c_3 = a_2$),

$$p_1 = b_1 p_{2xy} + b_2 p_{xx} + c_1 \left(\int p_{2y} dx \right)^2 + c_2 p_2 \int p_{2y} dx + c_3 p_2^2. \quad (70)$$

From (11) and (70) we know

$$u_1 = -2b_1 v_{xy} - 2b_2 v_{xx} + 4c_1 \left(\int v_y dx \right)^2 + 4c_2 v \left(\int v_y dx \right) + 4c_3 v^2 \quad (71)$$

and the related Lax pair becomes

$$\psi_{xx} = v\psi, \quad (72)$$

$$\begin{aligned} \psi_t = & (4c_1 \left(\int v_y dx \right)^2 + 4c_2 v \left(\int v_y dx \right) + 4c_3 v^2 - 2b_1 v_{xy} - 2b_2 v_{xx}) \psi_x \\ & + (\lambda_1 - 2v_x(2c_3 v + c_2 \int v_y dx) - 2v_y(2c_1 \int v_y dx + c_2 v) + b_1 v_{xxy} + b_2 v_{xxx}) \psi. \end{aligned} \quad (73)$$

while the corresponding evolution for the field v is

$$\begin{aligned} v_t = & b_1 v_{xxxxxy} + b_2 v_{xxxxxx} + 2v_2(4c_2 v^2 - 6c_1 v_{xy} - 3c_2 v_{xx} + 8c_1 v \int v_y dx) \\ & - 2(c_2 v + 2b_1 v + 2c_1 \int v_y dx) v_{xy} - 2(2c_3 v + 2b_2 v + c_2 \int v_y dx) v_{xx} \\ & + 2(10c_3 v^2 + 2c_1 \left(\int v_y dx \right)^2 6c_2 v \int v_y dx - (b_2 + 6c_3) v_{xx} - (b_1 + 3c_2) v_{xy}). \end{aligned} \quad (74)$$

It is obvious that when $y = x$ and/or $v_y = 0$, (2+1)-dimensional fifth order equation (74) will be reduced back to (1+1)-dimensional FOKdV equation (45).

5 On spectral parameters

In the last two sections, we have omitted the spectral parameter(s). In order to add the possible spectral parameter(s) to the Lax pairs, we may use the symmetry transformations of the original nonlinear models. In some cases, to find a symmetry transformation such that a nontrivial parameter can be included in the Lax pair (8) and (9) is quite easy. For instance, it is well known that the KdV equation (28) is invariant under the Galileo transformation

$$u_1 \rightarrow u_1(x + 3\lambda t, t) + \lambda \equiv u_1(x', t) + \lambda. \quad (75)$$

Substituting (75) into (26) and (27) yields the usual Lax pair of the KdV equation with spectral parameter λ :

$$\psi_{xx} - \frac{1}{2}(u_1 + \lambda)\psi = 0, \quad (76)$$

$$\psi_t = (u_1 - 2\lambda)\psi_x - \frac{1}{2}u_{1x}\psi, \quad (77)$$

where x' has been rewritten as x .

However, for some other models to add the parameters to (8) and (9) is quite difficult. In other words, the spectral parameters may be included in (8) and (9) in very complicated way(s). For instance, for the CDGSK equation ((45) with (47)), we failed to include a nontrivial spectral parameter by using its point Lie symmetries. Nevertheless, if we use the higher order symmetries and/or nonlocal symmetries of the model, we can include some nontrivial parameters in (43) and (44) with (47). For instance, for the CDGSK equation, if ψ_1 is a special solution of (43) and (44) with (47), one can prove that

$$u' = u - 6 \frac{\lambda(\lambda\psi_1^2 - \lambda\psi_{1x}p - 6\psi_{1x})}{(\lambda p + 6)^2} \quad (78)$$

with

$$p_x = \psi_1 \quad (79)$$

is also a solution of the CDGSK equation. By substituting (78) into (43) and (44), we obtain a second order Lax pair ($P = 6 + \lambda p$)

$$\begin{aligned} \psi_{xx} &= - \left(u - 6 \frac{\lambda(\lambda\psi_1^2 - \lambda\psi_{1x}p - 6\psi_{1x})}{(\lambda p + 6)^2} \right) \psi, \\ \psi_t &= \left(\frac{6\lambda(u_x\psi_1)_x}{P} - 12 \frac{\psi_1\lambda^2(3u\psi_{1x} + 2u_x\psi_1)}{P^2} + 72\lambda^3\psi_1 \frac{u\psi_1^2 - \psi_{1x}^2}{P^3} \right) \psi, \end{aligned} \quad (80)$$

$$\begin{aligned}
& -u_{xxx} - ww_x - 36\lambda^4\psi_1^3 \frac{2\lambda\psi_1^2 - 5\psi_{1x}P}{P^5} \Big) \psi \\
& + \left(36\lambda^2 u \frac{\psi_1^2}{P^2} + 2u_{xx} + u^2 - 12\lambda\psi_{1x} \frac{u_x}{P} - 36\lambda^4 \frac{\psi_1^4}{P^4} + 72\lambda^3\psi_1^2 \frac{\psi_{1x}}{P^3} \right) \psi_x \quad (81)
\end{aligned}$$

for the CDGSK model with ψ_1 being a solution of (43) and (44).

6 Summary and discussions

In summary, every (1+1)-dimensional equation which has a Schwarz variant may possess a second order Lax pair. In this paper, we prove the conclusion when the Schwarz form is an arbitrary function of five conformal invariants and their any order derivatives and integrations.

Usually, the Lax operators for various integrable models (except for the KdV hierarchy) are taken as higher order operator. Though the order of the Lax pair operators for some models have been lower down, the spectral parameter have been disappeared. In order to recover some types of nontrivial spectral parameters, we have to use the symmetries of the original nonlinear equations and the spectral parameter(s) would be appeared in the second order Lax operator in some complicated ways. How to obtain some other integrabilities from the Lax pairs listed here for general or special models is worthy of study further though the spectral parameters have not yet been included in explicitly. One may obtain many interesting properties of some special models from the Lax pairs without spectral parameters^[15, 16]. For instance, infinitely many nonlocal symmetries of the KdV equation, HD equation, CDGSK equation and the KK equation can be obtained from the spectral parameter independent Lax pairs^[15].

The conclusion for the general (1+1)-dimensional Schwarz equations can also be extended to some special types of (2+1)-dimensional models, like the breaking soliton equations. However, how to extend the method and the conclusions to general (2+1)-dimensions or even in higher dimensions is still open.

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